tion assembly shown in Fig. 5 can be used. Although this assembly generates slightly larger odd harmonics it is generally superior to that of Fig. 1 because it does not generate even harmonics. It can be shown that

$$E_{\text{reflected}} = E_{\text{in}} \sin^2 \phi$$

where

$$\phi = (\pi/4 + \theta_{\max} \sin \omega_m t)$$

and that this is identical to

$$E_{\text{reflected}} = \frac{E_{\text{in}}}{2} [1 + \sin (2\theta_{\max} \sin \omega_m t)].$$

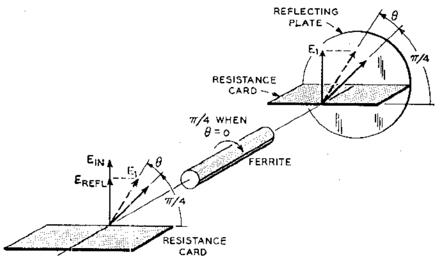


Fig. 5—A reflecting microwave modulator which does not generate even harmonics.

Again using (1) of Rizzi and Rich[2] it can be shown that the reflected voltage varies only at the fundamental and odd harmonic modulation rates.

$$E_{\text{reflected}} = E_{\text{in}}[1/2 + J_1(2\theta_{\max}) \sin \omega_m t + J_3(2\theta_{\max}) \sin 3\omega_m t + \cdots ].$$

Thus, it appears that Mr. Clavin is in error when he claims that a strong second harmonic will result from a reflected voltage which varies as the $\sin^2 \phi$.

An analysis of this last equation yields data identical to that of Fig. 2 of Rizzi and Rich.[2] This figure shows how the largest harmonic coefficients, the third and fifth, vary with respect to the fundamental coefficient for different maximum values of $2\theta_{\max}$. In particular this figure shows that the third and fifth harmonics are down more than 36 db and 55 db respectively from the fundamental when the maximum value of $2\theta_{\max}$ is less than or equal to 36°, the value required to again reduce the fundamental coefficient, $J_1(2\theta_{\max})E_{\text{in}}$ to the same value of 0.3 $E_{\text{in}}$ which corresponds to a 60 per cent modulation of a linear system.

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## On Symmetrical Matching*

Mr. Mathis' note[1] is correct in that a match is achieved with the three shunt susceptances, but he is wrong in his assertion that no other positioning is possible for match. His method of matching is outlined in Fig. 1.

* Received by the PGMTT, November 13, 1956.
[1] H. F. Mathis, IRE TRANS., vol. MTT-4, p. 132; April, 1956.

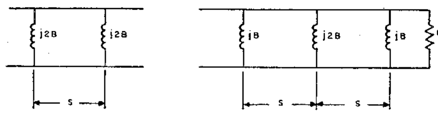


Fig. 1.

The distance $s$ is computed so that two shunt susceptances, each of magnitude $2B$ will cancel each other,[2] thus $\tan 2\pi s/\lambda g = 1/B$. Then susceptances of magnitude $B$ are spaced this distance away on either side and match is obtained. However, match is also obtained when $\tan 2\pi s/\lambda g = 2/B$ and thus the response is not symmetrical about the original frequency. The match can be checked by computing the admittance at the center of the network with a matched termination and, if it is purely real, the network is matched.[3]

The response can be made to be symmetrical (critically coupled) provided that the standing-wave ratios introduced by the three susceptances go in the ratios of $r$, $r^2$, $r$ or the susceptances go as $B$, $B\sqrt{B^2+4}$, $B$. The spacing $p$ between the three susceptances can be found on a Smith chart. See Fig. 2.

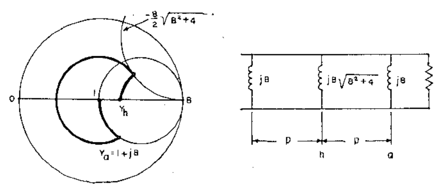


Fig. 2.

A matched termination at the right of the line makes the normalized admittance at point $a$ $1+jB$. In Fig. 2, inductive susceptances which will have negative values are assumed. This admittance is transformed along the line toward the generator until the circle is tangent to the Smith chart circle for

$$-\frac{B}{2}\sqrt{B^2+4}.$$

The admittance at the center of the network then will be purely real and chart of the network will be the mirror image about the real axis. Therefore, the input admittance is matched. For small variations in length of line $p$ corresponding to small changes in frequency the input admittance is still matched since the circles are tangent. This circuit is thus a critically-coupled double tuned arrangement. The value of the line length $p$ is given by the formula.

$$\tan \frac{2\pi p}{\lambda g} = \frac{B^2 + 2 + \sqrt{B^2+4}}{B^3 + 3B}.$$

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[2] J. Reed, "Low-Q microwave filters," PROC. IRE, vol. 38, p. 794; July, 1950.
[3] *Ibid.* See (6).

### *Author's Comment*[4]

Mr. Reed is correct in his remarks. My note was based on the theorem that *if one-sided matching for a lossless symmetrical discontinuity is achieved with a lossless symmetrical matching network, the matching network can be split and the part farther from the discontinuity moved to the opposite side of the discontinuity to obtain two-sided matching.* This theorem is correct, but all of the conclusions in my note were not correct.

In general, either the value of the shunt susceptances or their positions for symmetrical matching may be arbitrarily selected. When the value of the shunt susceptances is selected, there may be two pairs of positions which can be used. When the positions are selected, there may be two values of the shunt susceptances which can be used.

Procedures for finding the positions or the value of the shunt susceptances, when the other is given, are presented next. In the discussion which follows, it is assumed that the voltages, currents, and impedances are measured in units so that the characteristic impedance of the transmission line is one.

When the value $B$ of the shunt susceptances is selected, the discontinuity is terminated in a matched load, and the input admittance $Y_1$ is determined and plotted on a Smith chart, as shown in Fig. 3. The admittance $Y_2$ given by the formula

$$Y_2 = \frac{16 + 12B^2 + 3B^4 - j(16B + 8B^3)}{16 + 12B^2 + B^4} \quad (1)$$

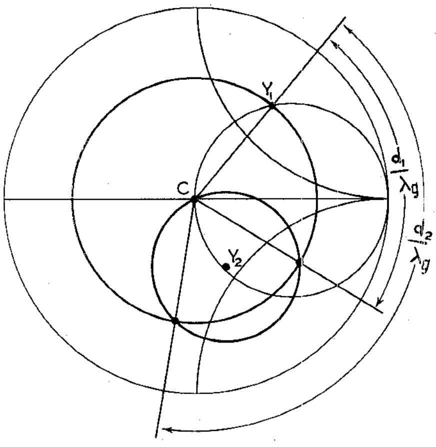


Fig. 3—Diagram for determining the positions of shunt susceptances.

is plotted on the Smith chart. A circle is drawn with its center at $Y_2$ which passes through the center $C$ of the chart. (This circle must also pass through the point $Y = 1 - j2B$.) A circle is drawn with its center at $C$ which passes through the point $Y_1$. The points of intersection of these circles determine the possible pairs of positions of the shunt susceptances. (For the second example given by Reed, the circles are tangent.) If the circles do not intersect, it is not possible to use this value of $B$. The distances $d_1$ and $d_2$ of these pairs of positions

[4] Received by the PGMTT, November 23, 1956.

from the discontinuity are determined on the Smith chart as indicated in Fig. 3.

When the positions of the shunt susceptances are selected, the input admittance $g+jb$ at one of the positions is determined with the opposite side of the discontinuity terminated in a matched load. The possible values $B$ of the shunt susceptances are given by the formula

$$B = \frac{b \pm \sqrt{gb^2 + (g-1)^3}}{g-1}. \tag{2}$$

If $B$ is not real, it is not possible to use these positions.

### Proof of Theorem

According to well-known circuit theory, any lossless four-terminal network can be represented by the matrix

$$\begin{pmatrix} A_{1,1} & jA_{1,2} \\ jA_{2,1} & A_{2,2} \end{pmatrix},$$

where the input voltage $E_1$ and the input current $I_1$ are related to the output voltage $E_2$ and the output current $I_2$ by the equations

$$E_1 = A_{1,1}E_2 + jA_{1,2}I_2,$$
$$I_1 = jA_{2,1}E_2 + A_{2,2}I_2.$$

The symbols $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, and $A_{2,2}$ denote real quantities. If the network is symmetrical, $A_{1,1}=A_{2,2}$. If the network is reversed, the corresponding matrix is

$$\begin{pmatrix} A_{2,2} & jA_{1,2} \\ jA_{2,1} & A_{1,1} \end{pmatrix}.$$

If the input impedance is one when the load impedance is one, then $A_{1,1}=A_{2,2}$ and $A_{1,2}=A_{2,1}$. A section of lossless transmission line, with a characteristic impedance of one, is represented by the matrix

$$\begin{pmatrix} \cos\beta d & j\sin\beta d \\ j\sin\beta d & \cos\beta d \end{pmatrix}.$$

Let the symmetrical lossless discontinuity be represented by the matrix

$$\begin{pmatrix} M & jN \\ jP & M \end{pmatrix}.$$

The symmetrical one-sided lossless matching network is split into two networks which are represented by the matrices

$$\begin{pmatrix} m & jn \\ jp & q \end{pmatrix}$$

and

$$\begin{pmatrix} r & js \\ jt & u \end{pmatrix},$$

The matrix for the three networks connected in cascade is

$$\begin{pmatrix} A_{1,1} & jA_{1,2} \\ jA_{2,1} & A_{2,2} \end{pmatrix} = \begin{pmatrix} m & jn \\ jp & q \end{pmatrix}\begin{pmatrix} r & js \\ jt & u \end{pmatrix}\begin{pmatrix} M & jN \\ jP & M \end{pmatrix}.$$

Since the matching network is symmetrical,

$$mr - nt = qu - ps.$$

Also, since the input impedance is one when the load impedance is one,

$$\begin{aligned} A_{1,1} &= A_{2,2} \\ &= (mr - nt)M - (ms + nu)P \\ &= (qw - ps)M - (pr + qt)N \end{aligned}$$

and

$$\begin{aligned} A_{1,2} &= A_{2,1} \\ &= (ms + nu)M + (mr - nt)N \\ &= (pr + qt)M + (qu - ps)P. \end{aligned}$$

If the part of the matching network farther from the discontinuity is moved to the opposite side of the discontinuity, the matrix for the resulting network is

$$\begin{pmatrix} B_{1,1} & jB_{1,2} \\ jB_{2,1} & B_{2,2} \end{pmatrix} = \begin{pmatrix} r & js \\ jt & u \end{pmatrix}\begin{pmatrix} M & jN \\ jP & M \end{pmatrix}\begin{pmatrix} m & jn \\ jp & q \end{pmatrix}.$$

The values of $B_{1,1}$, $B_{1,2}$, $B_{2,1}$, and $B_{2,2}$ are given by the equations

$$B_{1,1} = (mr - ps)M - rtN - puP,$$
$$B_{1,2} = (nr + qs)M + qrN - nsP,$$
$$B_{2,1} = (mt + pu)M - ptN + muP,$$
$$B_{2,2} = (qu - nt)M - qtN - nuP.$$

By simple algebraic manipulations, it can be shown that the above equations require that $B_{1,1}=B_{2,2}$ and $B_{1,2}=B_{2,1}$. Consequently, if the load impedance is one, the input impedance is one. This completes the proof of the theorem.

Next, symmetrical matching is considered. In this case, the matrix for the combined network is

$$\begin{pmatrix} C_{1,1} & jC_{1,2} \\ jC_{2,1} & C_{2,2} \end{pmatrix} = \begin{pmatrix} m & jn \\ jp & q \end{pmatrix}\begin{pmatrix} M & jN \\ jP & M \end{pmatrix}\begin{pmatrix} q & jn \\ jp & m \end{pmatrix}.$$

Since $C_{1,2}=C_{2,1}$,

$$2mnM + m^2N - n^2C = 2pqM - p^2N + q^2P.$$

If the right-hand network is moved to the left with a section of lossless transmission line between the two matching networks, the corresponding matrix is

$$\begin{pmatrix} D_{1,1} & jD_{1,2} \\ jD_{2,1} & D_{2,2} \end{pmatrix} = \begin{pmatrix} q & jn \\ jp & m \end{pmatrix} \cdot \begin{pmatrix} \cos\beta d & j\sin\beta d \\ j\sin\beta d & \cos\beta d \end{pmatrix}\begin{pmatrix} m & jn \\ jp & q \end{pmatrix}\begin{pmatrix} M & N \\ jP & M \end{pmatrix}.$$

If $\beta d$ is chosen so that $D_{1,1}=D_{2,2}$, then it can be shown by simple algebraic manipulations that $D_{1,2}=D_{2,1}$. Consequently, if symmetrical two-sided matching can be achieved, there exists a corresponding one-sided symmetrical matching network. So the procedures given above yield all possible symmetrical matching networks of the type considered.

### Derivation of Procedures

According to the above theorem, if $Y_A=1$ for the circuit shown in Fig. 4, then $Y_B=1$ for the circuit shown in Fig. 5. It may be observed that the dimension $d_3$ does not appear in Fig. 5.

For the circuit shown in Fig. 4, let $Y_1$ denote the admittance at the input of the loss ess discontinuity, $Y_a$ denote the

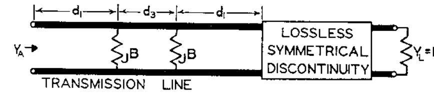


Fig. 4—One-sided matching.

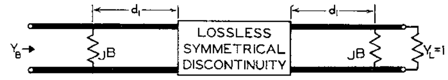


Fig. 5—Two-sided matching.

admittance immediately to the right of the right shunt susceptance, $Y_b$ denote the admittance immediately to the left of the right shunt susceptance, and $Y_c$ denote the admittance immediately to the right of the left shunt susceptance. Let $Y_a=g+jb$. Now $Y_b=g+j(b+B)$. In order for the left shunt susceptance to complete the match, $Y_c=1-jB$. Three sets of these admittances are shown plotted on Smith charts in Figs. 6 and 7. (For these illustrations, the discontinuity consists of a shunt susceptance whose value is 1.6.)

The point $Y_b$ must lie on the circle $C_1$ through $Y_c$ whose center is at $C$. Let $C_2$ denote the circle which is obtained by subtracting $jB$ from the admittances on the circle $C_1$. Since $Y=1+jB$ lies on $C_1$, $Y=1$ must lie on $C_2$, *i.e.*, the circle $C_2$ must pass through $C$. The point $Y_a$ must lie on $C_2$. The point $Y_a$ must also lie on the circle $C_3$ through $Y_1$ whose center is at $C$. The distances $d_1$ and $d_3$ are indicated by the angles $\angle Y_1CY_a$ and $\angle Y_bCY_c$, respectively.

For a given value of $B$, the circle $C_2$ is fixed. In general, the circle $C_3$ intersects the circle $C_2$ at two points. This is illustrated in Figs. 3 and 6. The value of $B$ is the same for Figs. 6(a) and (b), but $d_1$ and $d_3$ are different. (For Fig. 6, $Y_1=1+j1.6$ and $B=1$.)

The circle $C_2$ can be drawn by finding at least three points on it by subtracting $jB$ from the admittances for points on the circle $C_1$, and then drawing a circle through these points. However, it is more convenient to compute the center of the circle. In the $\Gamma$-plane, the center of the circle is at $\Gamma=0$ and the radius of this circle is

$$\frac{B}{\sqrt{4+B^2}}.$$

The bilinear transformation[5]

$$Y = \frac{1+\Gamma}{1-\Gamma}$$

$Y$ plane. The center of $C_4$ is at

$$Y = \frac{2+B^2}{2}$$

and the radius of $C_4$ is

$$\frac{B\sqrt{4+B^2}}{2}.$$

The circle $C_5$ in the $Y$ plane is found by subtracting $jB$ from the admittances on $C_4$. The center of $C_5$ is at

[5] H. F. Mathis, "Bilinear transformations," IRE Trans., vol. CT-3, p. 156; June, 1956.

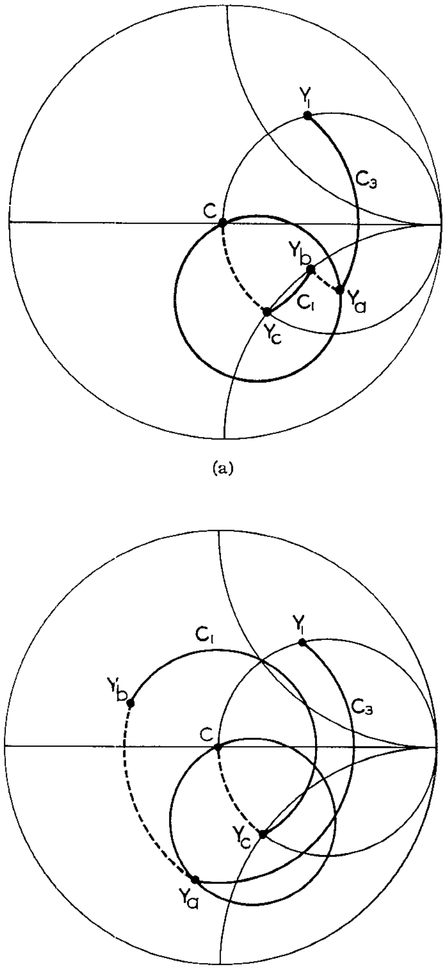


Fig. 6—Diagrams illustrating matching with fixed $B$.

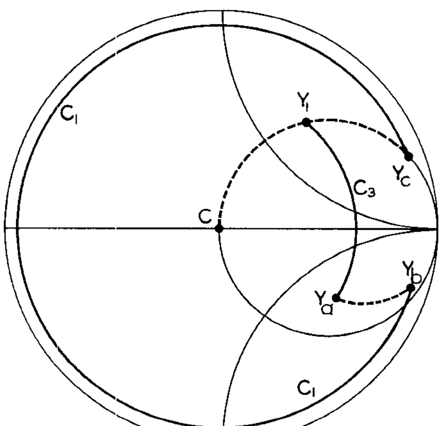


Fig. 7—Diagram illustrating matching with fixed $d_1$.

$$Y = \frac{2+B^2}{2} - jB$$

and the radius of $C_5$ is the same as the radius of $C_4$. The transformation

$$\Gamma = \frac{Y-1}{Y+1}$$

maps the circle $C_5$ into the circle $C_2$ in the $\Gamma$ plane. The center of the circle $C_2$ is at

$$= \frac{B^2 - i2B}{4+2B^2}.$$

The corresponding admittance of the center of $C_2$ is

$$Y_2 = \frac{1+\Gamma}{1-\Gamma}$$

$$= \frac{16+12B^2+3B^4-j(16B+8B^3)}{16+12B^2+B^4}. \quad (1)$$

Let $\Gamma_b$ and $\Gamma_c$ denote the current reflection coefficients corresponding to $Y_b$ and $Y_c$, respectively. The values of $\Gamma_b$ and $\Gamma_c$ are given by the equations

$$\Gamma_b = \frac{g+j(b+B)-1}{g+j(b+B)+1},$$

$$\Gamma_c = \frac{1-jB-1}{1-jB+1}.$$

Since $Y_b$ and $Y_c$ lie on the circle $C_1$,

$$|\Gamma_b|^2 = |\Gamma_c|^2 = \frac{(g-1)^2+(b+B)^2}{(g+1)^2+(b+B)^2}$$

$$= \frac{B^2}{2^2+B^2}.$$

This equation can be solved for $B$ to obtain (2). If $g=1$, it is obvious that $2B=-b$. Thus for a given $d_1$ and the corresponding $Y_a=g+jb$, there are generally two possible values of $B$. This is illustrated in Figs. 6(a) and 7 where the value of $Y_a$ is the same but the values of $B$ are different. [For Fig. 6(a), $Y_a=1.9-j2$ and $B=1$. For Fig. 7, $B=-5.4$.]

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## *Letter from Mr. Reed*[6]

Mr. Mathis' theorem is correct but the procedure resulting from this theorem does not give a good result from an engineering standpoint. The result of what he calls two-sided matching will give a match not only at the design center frequency, but also at some other frequency. Thus, the performance curve will *not* be symmetrical about the design center. The procedure suggested in my last note would give a symmetrical curve with maximally-flat response in which the two frequencies of match are the same.

Suppose it is desired to cancel out an inductive iris which has a normalized susceptance of $-2$. The reflection from this can be cancelled out by the use of another iris whose susceptance is also $-2$ spaced down the line toward the generator by three-eighths of a wavelength.

Thus, according to his theorem, we can split the matching into two susceptances of $-1$ on either side of the susceptance of $-2$ spaced 0.375 $\lambda g$ (tan $2\pi s/\lambda g=-1$) of a wavelength from it. But match would also occur if the spacing were 0.3245 $\lambda g$(tan $2\pi s/\lambda g=-2$).

For critical coupling $B\sqrt{B^2+4}$ is set equal to $-2$ and the resulting equation solved for $B$ giving $B$ to $-0.91018$. This value of $B$ is inserted into the formula for $p$, thus resulting in this case of a value of $p$ equal to $0.3465\lambda g$. See Fig. 8.

[6] Received by the PGMTT, December 19, 1956.

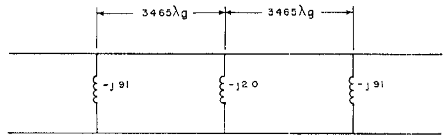


Fig. 8.

Approximate performance curves for one-sided matching, two-sided matching using the Mathis theory, and critically coupled performance as described above are shown below. Some improvements in the critically coupled performance can be obtained by letting the midband be mismatched but be matched on either side of the design frequency. See Fig. 9 below.

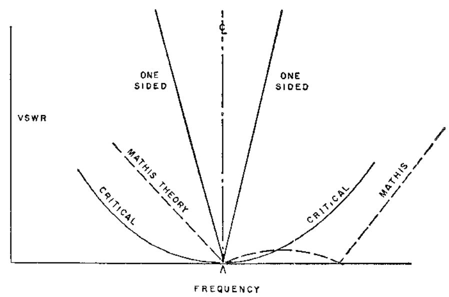


Fig. 9.

JOHN REED

## *Author's Comment*[7]

I agree with the remarks in Mr. Reed's recent note. In my original brief notes, I did not consider the effects of varying the frequency. His two notes are most interesting and valuable. I do not think that I can add anything of value.

H. F. MATHIS

[7] Received by the PGMTT, January 27, 1957.

# The Available Power of a Matched Generator from the Measured Load Power in the Presence of Small Dissipation and Mismatch of the Connecting Network*

It is sometimes necessary to determine the available power of a matched generator in terms of the power dissipated in a load when the load is connected to the generator by means of a slightly mismatched 4-pole having small loss. (A piece of waveguide or short length of coaxial line could exemplify such a 4-pole; the discontinuities at flanges or at connectors and supporting beads could give rise to the slight mismatch.)

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